

MODELLING ROTATIONAL ELASTICITY WITH ORTHOGONAL MATRICES

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ABSTRACT. It is the aim of the paper to present a new point of view on rotational elasticity in a nonlinear setting using orthogonal matrices. The proposed model, in the linear approximation, reduces to the well known equilibrium equations of static linear elasticity. An appropriate kinetic energy is identified and we present a dynamical model of rotational elasticity. The propagation of elastic waves in such a medium is studied and we find two classes of waves, transversal rotational waves and longitudinal rotational waves, both of which are solutions of the nonlinear partial differential equations. For certain parameter choices, the transversal wave velocity can be greater than the longitudinal wave velocity. Moreover, parameter ranges are identified where the model describes an auxetic material. However, in all cases the potential energy functional is positive definite. We present one possible way of visualising transversal rotational waves.

1. INTRODUCTION

We are developing a theory of rotational elasticity based on using orthogonal matrices as the basic dynamical variables. Let us consider a three-dimensional elastic continuum, occupying an open connected set $\Omega \subset \mathbb{R}^3$ or the whole of \mathbb{R}^3 , whose material points can only experience rotations and no displacements. Our Euclidean space is assumed to be equipped with Cartesian coordinates $\mathbf{x} = (x, y, z)$. To every point $\mathbf{x} \in \mathbb{R}^3$ we attach an orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. In the initial state these basis vectors are assumed to be aligned with our Cartesian coordinate axes, i.e. $(\mathbf{e}_i)_j = \delta_{ij}$, but will become functions of \mathbf{x} and time t once the continuum is deformed.

Models of this type have in fact a long tradition. One such model has first been introduced by MacCullagh in 1839, see [27]. He noted that it was not possible to describe optical phenomena by comparing the aether with an ordinary elastic solid. He thus introduced a new type of medium whose potential energy depended only on rotations. Similar models have been investigated by the Cosserat brothers which resulted in an extended framework of elasticity [5], often referred to as Cosserat elasticity. The main difference between Cosserat elasticity and classical elasticity is the assumed independence of displacements and rotations, often referred to as microrotations. When formulating an energy functional of classical elasticity, one assumes the integrand not to depend on the derivatives of rotation. It is well known that rotations about different axes do not commute, and thus we expect an inherently nonlinear theory. Variants of the theory of Cosserat elasticity appear under various names in modern applied mathematics literature such as oriented

medium, asymmetric elasticity, micropolar elasticity to mention a few popular ones, see e.g. [7, 6, 21, 26, 13, 10, 11, 23, 8].

This field is also related to the theory of granular media, ferromagnetic materials, cracked media and liquid crystals to mention a few of them. There has also been an interest in the theory of Cosserat elasticity from a more theoretical physics point of view, see [24, 4, 18, 17].

In this work we are only considering rotational deformations of the continuum and neglect displacements. We assume that every material point can experiences an independent rotation $\mathbf{e}^i \mapsto \mathbf{O}(\mathbf{x})\mathbf{e}^i$, $i = 1, 2, 3$, where $\mathbf{O}(\mathbf{x})$ denotes the field of orthogonal matrices, $\mathbf{O}^T \mathbf{O} = \mathbf{O} \mathbf{O}^T = \mathbf{I}$ where \mathbf{O}^T denotes the transpose and \mathbf{I} the identity matrix. We consider orthogonal matrices with $\det \mathbf{O} = +1$. The most general potential energy functional of such a static continuum is thus taken to be

$$(1.1) \quad V := \int_{\Omega} f(\mathbf{x}, \mathbf{O}, \partial \mathbf{O}, \partial^2 \mathbf{O}, \dots) dx dy dz$$

where $\partial \mathbf{O}$ ($\partial^2 \mathbf{O}$, ...) denotes the first (second, ...) partial derivatives of \mathbf{O} . The matrix-function $\mathbf{O}(\mathbf{x})$ is the unknown quantity, our dynamical variable.

2. STATEMENT OF THE STATIC PROBLEM

2.1. Basic equations of the static model. The general framework (1.1) of our rotational elasticity model is very wide. Hence, as a first step we identify a suitable subclass of variational functionals which is of mechanical and physical interest, and which is invariant under rigid rotations. This means the mapping $\mathbf{O} \mapsto \mathbf{O} \bar{\mathbf{O}}$, where $\bar{\mathbf{O}}$ is a constant orthogonal matrix, leaves the variational functional invariant.

Possible deformations of our elastic medium are characterised by the quantity

$$\mathbf{K}_{ijk} := \mathbf{O}_{im} \partial_j \mathbf{O}_{km}$$

where we sum over repeated indices. In the context of Cosserat elasticity, this quantity is sometimes referred to as contortion [18, 17]. Note that \mathbf{K}_{ijk} is, by definition, skew-symmetric in the first and third index, $\mathbf{K}_{ijk} = -\mathbf{K}_{kji}$. This follows from the fact that $\partial(\mathbf{O} \mathbf{O}^T) = 0$. Moreover, the quantity \mathbf{K} is invariant under rigid rotations. Since we are working in \mathbb{R}^3 it seems natural to contract \mathbf{K}_{ijk} with the Levi-Civita symbol to construct a 3×3 matrix to analyse the deformations, we define

$$\mathbf{A}_{mn} := \varepsilon_{mjl} \mathbf{K}_{jnl}.$$

Since the matrix \mathbf{A} has no a priori symmetries, it is useful to decompose it into its three irreducible components, a trace part, a skew-symmetric part and a trace-free symmetric piece, respectively defined by

$$\begin{aligned} \mathbf{A}_{mn}^{(1)} &:= \frac{1}{3} \mathbf{A}_{ii} \mathbf{I}_{mn} && \text{trace} \\ \mathbf{A}_{mn}^{(2)} &:= \frac{1}{2} (\mathbf{A}_{mn} - \mathbf{A}_{nm}) && \text{skew - symmetric} \\ \mathbf{A}_{mn}^{(3)} &:= \mathbf{A}_{mn} - \mathbf{A}_{mn}^{(1)} - \mathbf{A}_{mn}^{(2)} && \text{trace - free symmetric.} \end{aligned}$$

Note that the definition for $\mathbf{A}_{mn}^{(3)}$ can also be written in the form

$$\mathbf{A}_{mn}^{(3)} = \frac{1}{2} (\mathbf{A}_{mn} + \mathbf{A}_{nm}) - \frac{1}{3} \mathbf{A}_{ii} \mathbf{I}_{mn}$$

which makes explicit that this piece is indeed trace-free and symmetric.

This trace-free symmetric piece is closely related to the so-called Q -tensor of the Landau-de Gennes theory of uniaxial nematic liquid crystals, see [2, 20]. The Q -tensor is symmetric, traceless and in addition has two equal eigenvalues. The elastic energy density in this theory is based on elastic invariants constructed from the irreducible components of the first partial derivatives of Q . In our model this corresponds to $\partial^2 \mathbf{O}$, the second partial derivatives of \mathbf{O} . Therefore, our general potential energy (1.1) is sufficiently wide to cover a large variety of materials.

A natural starting point would be to consider a functional of the type

$$V_1 = \int_{\Omega} \left[c_1 \|\mathbf{A}^{(1)}\|^2 + c_2 \|\mathbf{A}^{(2)}\|^2 + c_3 \|\mathbf{A}^{(3)}\|^2 \right] dx dy dz$$

where c_1 , c_2 and c_3 are positive constants which are naturally called the *elastic moduli*. A similar functional was considered in [24] to formulate a unified field theory for mesons. Moreover, we denote the norm by $\|\mathbf{A}^{(i)}\|^2 = \mathbf{A}_{mn}^{(i)} \mathbf{A}_{mn}^{(i)}$ where $i = 1, 2, 3$ and no summation over i . Note that a similar functional has been considered in the context of Cosserat elasticity [14]. The three elastic moduli are not fully independent when one works in the whole of \mathbb{R}^3 because of the following identity. One can show that

$$(2.1) \quad -2\|\mathbf{A}^{(1)}\|^2 - \|\mathbf{A}^{(2)}\|^2 + \|\mathbf{A}^{(3)}\|^2 = 4\varepsilon_{ijk} \partial_i \mathbf{A}_{jk}^{(2)}.$$

This identity has its roots in differential geometry. Of course, when Ω has a boundary, the above argument fails and the identity (2.1) leads to the appearance of boundary terms. One can regard \mathbf{K} as an affine connection on a globally flat Riemannian manifold with curvature and torsion¹. The definition of the Riemann curvature tensor involves derivatives of the connection and contraction of the connection with itself. The vanishing of the Riemann curvature tensor is then equivalent to (2.1). Note that the vanishing of the Riemann curvature tensor is also often used in classical elasticity since one assumes that the distorted medium remains flat, thus implying certain compatibility conditions.

Next we can rewrite $\|\mathbf{A}^{(3)}\|^2$ in the energy functional V_1 in terms of the other two terms and a surface term which will not alter the resulting equations of motions. This results in the functional

$$(2.2) \quad V_2 = \int_{\Omega} \left[\hat{c}_1 \|\mathbf{A}^{(1)}\|^2 + \hat{c}_2 \|\mathbf{A}^{(2)}\|^2 \right] dx dy dz$$

where $\hat{c}_1 = c_1 + 2c_3$ and $\hat{c}_2 = c_2 + c_3$.

2.2. Basic equations of the linearised model. It is known that every orthogonal matrix \mathbf{O} can be written as the exponential of a skew-symmetric matrix. It turns out to be convenient to analyse the nonlinear equations and their relation with linear elasticity by writing $\mathbf{O} = \exp(\mathbf{u}^*)$. We denote by \mathbf{u}^* the a skew-symmetric

¹Alternatively, one can work with the torsion tensor \mathbf{T} directly [3], which is defined by $T_{jkl} := K_{jkl} - K_{jlk}$. This tensor can also be contracted with the Levi-Civita symbol to construct a 3×3 matrix to be decomposed into irreducible pieces. Both formulations are equivalent and can easily be translated into each other by noting that $\mathbf{A}^{(1)} = -T^{\text{ax}}$, $\mathbf{A}^{(2)} = -2T^{\text{vec}}$, $\mathbf{A}^{(3)} = 2T^{\text{ten}}$, $\|\dot{\mathbf{O}}\| = \sqrt{2}\|\omega\|$ and identifying $c_1 = c^{\text{ax}}$, $4c_2 = c^{\text{vec}}$, $4c_3 = c^{\text{ten}}$, $2\rho = c^{\text{kin}}$.

matrix dual to the vector \mathbf{u} . This means $(\mathbf{u}^*)_{ik} = \varepsilon_{ijk} u_j$ or, written out explicitly,

$$\mathbf{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}, \quad \mathbf{u}^* = \begin{pmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{pmatrix}.$$

In other words, rather than using elements of the group $\mathbf{O} \in \text{SO}(3)$ as the dynamical or state variables, we are using elements of its Lie algebra $\mathbf{u}^* \in \mathfrak{so}(3)$, see [14] where this approach is used in linear Cosserat elasticity.

Thus we can expand the orthogonal matrix in \mathbf{u} and obtain

$$(2.3) \quad \mathbf{O} = \exp(\mathbf{u}^*) = \mathbf{I} + \mathbf{u}^* + O(\|\mathbf{u}\|^2)$$

where $\|\mathbf{u}\|^2$ denotes the standard vector norm $\|\mathbf{u}\|^2 = u_x^2 + u_y^2 + u_z^2$. It should be noted that there exists a simple explicit representation for the matrix \mathbf{O} which is sometimes referred to as Rodrigues' formula

$$\exp(\mathbf{u}^*) = \mathbf{I} + \sin(\|\mathbf{u}\|) \frac{\mathbf{u}^*}{\|\mathbf{u}\|} + (1 - \cos(\|\mathbf{u}\|)) \frac{(\mathbf{u}^*)^2}{\|\mathbf{u}\|^2}.$$

Let us now assume \mathbf{u} to be small and let us expand the terms in the functional (2.2) up to terms $O(\|\mathbf{u}\|^3)$. This gives

$$\begin{aligned} \|\mathbf{A}^{(1)}\|^2 &= \frac{4}{3} \|\text{div } \mathbf{u}\|^2 + O(\|\mathbf{u}\|^3) \\ \|\mathbf{A}^{(2)}\|^2 &= 2 \|\text{curl } \mathbf{u}\|^2 + O(\|\mathbf{u}\|^3). \end{aligned}$$

Therefore, in the linear approximation the energy functional (2.2) becomes

$$(2.4) \quad V_3 = \int_{\Omega} \alpha \|\text{div } \mathbf{u}\|^2 + \beta \|\text{curl } \mathbf{u}\|^2 \, dx \, dy \, dz$$

where we denoted $\alpha = 4\hat{c}_1/3$ and $\beta = 2\hat{c}_2$.

We can also expand the term $\|\mathbf{A}^{(3)}\|^2$ in terms of the vector \mathbf{u} . However, it is more useful to consider the right-hand side of (2.1) which is given by

$$(2.5) \quad \varepsilon_{ijk} \partial_i \mathbf{A}_{jk}^{(2)} = \partial_m u_n \partial_n u_m - \|\text{div } \mathbf{u}\|^2 + O(\|\mathbf{u}\|^3).$$

Variation of the basic linear functional (2.4) with respect to our independent (dynamical) variable, the components of \mathbf{u}^* , yields

$$\delta V_3 = \int_{\Omega} \left(\frac{\partial V_3}{\partial \mathbf{u}_{ab}^*} \delta \mathbf{u}_{ab}^* + \frac{\partial V_3}{\partial (\partial_c \mathbf{u}_{ab}^*)} \delta (\partial_c \mathbf{u}_{ab}^*) \right) dx \, dy \, dz,$$

which after integration by parts and using Gauss' theorem gives

$$\delta V_3 = \int_{\Omega} \left(\frac{\partial V_3}{\partial \mathbf{u}_{ab}^*} - \partial_c \frac{\partial V_3}{\partial (\partial_c \mathbf{u}_{ab}^*)} \right) \delta \mathbf{u}_{ab}^* \, dx \, dy \, dz + \int_{\partial \Omega} n_c \frac{\partial V_3}{\partial (\partial_c \mathbf{u}_{ab}^*)} \delta (\partial_c \mathbf{u}_{ab}^*) \, dS.$$

To simplify matters, we assume Dirichlet boundary conditions and henceforth neglect the boundary term. This gives us the Euler-Lagrange equations

$$\alpha \text{grad}(\text{div } \mathbf{u}) - \beta \text{curl curl } \mathbf{u} = 0.$$

This system of equations has already been known by Cauchy and led to modifications due to MacCullagh and also Neumann, we refer the reader to [27].

The vector identity $\text{curl curl } \mathbf{u} = \text{grad}(\text{div } \mathbf{u}) - \Delta \mathbf{u}$ enables us to rewrite the Euler-Lagrange equations in the form

$$(2.6) \quad \beta \Delta \mathbf{u} + (\alpha - \beta) \text{grad}(\text{div } \mathbf{u}) = 0$$

Comparing with the well-known equilibrium equations of linear elasticity $\mu \Delta \mathbf{u} + (\mu + \lambda) \text{grad}(\text{div } \mathbf{u}) = 0$, we have to choose $\beta = \mu$ and $\alpha = \lambda + 2\mu$ to get a corresponding theory. Therefore, we have established an analogy between the linearised theory of the nonlinear rotational elasticity model and classical linear elasticity. Therefore, one can now argue that the energy functional (2.2) is in fact a ‘natural’ nonlinear generalisation of the functional of linear elasticity. Note that the equilibrium equations of linear elasticity are usually derived from a different functional, see [19, 22, 25]. If we denote $e_{ij} = (\partial_i u_j + \partial_j u_i)/2$, then the energy functional is given by

$$(2.7) \quad \int_{\Omega} \left[\frac{\lambda}{2} \|e_{ii}\|^2 + \mu \|e_{ij}\|^2 \right] dx dy dz.$$

It should be noted here that parts of the second term of the functional can be found in the right-hand side of the derivative term (2.5). Another interesting fact of linear elasticity is that not all the irreducible pieces of the matrix $\partial_i u_j$ are regarded as the fundamental building blocks of the potential energy functional. The skew-symmetric part, namely the rotations, are in general not part of the potential energy. It is the aforementioned Cosserat theory of elasticity which incorporates local rotations of points.

The functional (2.7) is obviously positive definite if the elastic moduli λ and μ are assumed to be positive. Similarly, the functional (2.4) is also positive definite provided that α and β are positive. Therefore, when comparing the nonlinear functional (2.2), which is positive definite provided $c_i > 0$ for $i = 1, 2, 3$, with the linear elasticity functional (2.7), we are led to identify

$$\lambda = 4 \left(\frac{1}{3} c_1 - c_2 - \frac{1}{3} c_3 \right), \quad \mu = 2(c_2 + c_3).$$

This immediately suggests that Poisson’s ratio for our rotational medium may not be between -1 and $1/2$ and, moreover, that the transverse wave velocity may be greater than the longitudinal wave velocity. By straightforward comparison we find for Poisson’s ratio σ and Young’s modulus E , respectively

$$\sigma = \frac{\lambda}{2(\lambda + \mu)} = \frac{c_1 - 3c_2 - c_3}{2c_1 - 3c_2 + c_3},$$

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} = \frac{12(c_1 - 2c_2)(c_2 + c_3)}{2c_1 - 3c_2 + c_3}.$$

Thus, by choosing the constants $c_i > 0$ appropriately, it is possible to define materials with negative Poisson’s ratio, also known as auxetic materials, see [16]. For ordinary materials with positive Young’s modulus $E > 0$ and positive Poisson’s ratio in the range $\sigma \in (0, 1/2)$, our positive constants have to satisfy $c_1 > 3c_2 + c_3$. An auxetic material with $\sigma \in (-1, 0)$ is realised when $2c_2 < c_1 < 3c_2 + c_3$.

Moreover, the constants $c_i > 0$ can also be chosen so that the analogous linearised functional (2.7) is not positive definite while the nonlinear theory in fact is. This fact is important when stability is investigated.

2.3. Coercivity and convexity. The equilibrium equations of our rotational elasticity theory are derived from an energy functional. Therefore, the nonlinear model is automatically a quasilinear, second order PDE system in divergence form [12]. A natural topic to address is the convexity of the potential energy [1]. For the linearised model (2.4) one can immediately verify that the functional is coercive,

independent of \mathbf{u} and convex in $\partial\mathbf{u}$. Thus, existence and uniqueness of the linearised Dirichlet boundary value problem are established in appropriate function spaces.

Since our basic variables of the nonlinear model are matrices, convexity of the functional will also depend on the convexity of the matrix exponential. The definition of a matrix valued convex function was introduced in [15] and is analogous to that of a scalar function. It follows that the matrix exponential is not convex in general. Our starting point functional is constructed to contain only sums of squares, so it is clearly bounded from below by zero.

However, coercivity of the functional remains a difficult task to be addressed since it does not seem to be true in general. By definition we have $\|\exp(\mathbf{u}^*)\|^2 = 3$, however, the norm $\|\mathbf{u}\|$ is *a priori* unconstrained.

2.4. Identifying kinetic energy. So far our model of rotational elasticity has been static. The next logical step is to introduce kinetic energy into the energy functional (2.2). To do this, we add a term of the form $\|\dot{\mathbf{O}}\|^2$ to the energy functional. To be consistent with our previous notation, we have

$$(2.8) \quad \|\dot{\mathbf{O}}\|^2 = \text{trace}(\dot{\mathbf{O}}\dot{\mathbf{O}}^T)$$

which is invariant under rigid rotations. If we now linearise this form of kinetic energy according to Eq. (2.3), we find

$$(2.9) \quad \|\dot{\mathbf{O}}\|^2 = 2\|\dot{\mathbf{u}}\|^2 + O(u^3).$$

Note that one regards $\|\dot{\mathbf{O}}\|$ as angular speed (modulus of the angular velocity vector) and $\rho\|\dot{\mathbf{O}}\|^2$ as rotational energy or angular kinetic energy, where ρ is the energy density of the medium.

2.5. Statement of the full problem. After identifying the kinetic energy, we can now formulate the complete variational functional of rotational elasticity in our setting.

Let us find a skew-symmetric matrix \mathbf{u}^* such that $\mathbf{O} = \exp(\mathbf{u}^*)$ extremises the variational functional or action

$$(2.10) \quad V = \int_{\Omega_T} \left[c_1 \|\mathbf{A}^{(1)}\|^2 + c_2 \|\mathbf{A}^{(2)}\|^2 + c_3 \|\mathbf{A}^{(3)}\|^2 - \rho \|\dot{\mathbf{O}}\|^2 \right] dx dy dz dt$$

where $\Omega_T := \Omega \times (0, T]$ with $T > 0$. We assume Dirichlet boundary conditions, namely $\mathbf{O} = \mathbf{I}$ on $\partial\Omega$. When Ω is the whole of \mathbb{R}^3 we assume $\mathbf{O} \rightarrow \mathbf{I}$ as $|\mathbf{x}| \rightarrow \infty$, which is equivalent to assuming $\mathbf{u}^* \rightarrow \mathbf{0}$ at the boundary. Note that in certain situations appropriate radiation conditions have to be taken into account.

3. PROPAGATION OF ROTATIONAL WAVES

3.1. Assumptions of the model. We are now discussing solutions to the fully nonlinear problem. As a first simplifying assumption, we consider a medium which can only experience rotation about one axis, the z -axis say. This means we can assume

$$(3.1) \quad \mathbf{u}^* = \varphi \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{O} = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\varphi = \varphi(x, y, z, t)$ is an arbitrary function of the three spatial variables and time. This choice for \mathbf{u}^* means that $u_x \equiv 0$ and $u_y \equiv 0$ in Eq. (2.3) and thus $\text{div } \mathbf{u} = \partial_x u_z$. To further simplify the problem, we follow the well-known approach of classical elasticity [19]. Note the interesting twist waves in liquid crystals analysed in [9].

Transversal rotational waves: We assume that the medium is homogeneous along the single axis of rotation, this means the z -axis. Thus, we restrict ourselves to $\varphi = \phi_t(x, y, t)$, which corresponds to the choice $\text{div } \mathbf{u} \equiv 0$.

Longitudinal rotational waves: By choosing φ to depend only on the z -coordinate and time, $\varphi = \phi_l(z, t)$, we effectively set $\text{curl } \mathbf{v} \equiv 0$.

We will use this approach to find rotational waves propagating through the elastic medium.

3.2. The Helmholtz equation. In general, since the matrix \mathbf{u}^* has three independent components, the variation with respect to its components yields three coupled second order PDEs. Using the above ansatz for the transversal and longitudinal rotational waves, two of the three equations are identically satisfied, and the remaining equations become

$$\begin{aligned} (c_2 + c_3)(\partial_{xx} + \partial_{zz})\phi_t - \rho\partial_{tt}\phi_t &= 0 & \text{transversal} \\ \frac{2}{3}(c_1 + 2c_3)\partial_{zz}\phi_l - \rho\partial_{tt}\phi_l &= 0 & \text{longitudinal} \end{aligned}$$

respectively. Note that these equations are linear, however, we did not linearise. It is our ansatz that makes the nonlinear terms disappear.

Therefore, in our elastic medium we find two types of rotational waves. Due to the nonlinear nature of the equations $\phi = \phi_t + \phi_l$ is not a solution of the Euler-Lagrange equations. As expected, the superposition principle does not hold.

The transversal rotational waves travel in the $x - y$ planes and are homogenous along the z -axis while the longitudinal rotational waves travel along the z -axis only. The corresponding wave velocities are given by

$$v_t = \sqrt{\frac{c_2 + c_3}{\rho}}, \quad v_l = \sqrt{\frac{2c_1 + 4c_3}{3\rho}}$$

with their ratio given by

$$(3.2) \quad \nu = \frac{v_t}{v_l} = \sqrt{\frac{3}{2} \frac{c_2 + c_3}{c_1 + 2c_3}}.$$

It is a well known fact in classical elasticity that the wave velocity of the longitudinal waves is always greater than the transversal wave velocity. This does not hold in our rotational elasticity model. For an ordinary material we found $c_1 > 3c_2 + c_3$. Inserting this into (3.2) the ratio of the velocities becomes

$$\nu = \sqrt{\frac{3}{2} \frac{c_2 + c_3}{c_1 + 2c_3}} < \sqrt{\frac{1}{2}}$$

which is in agreement with classical elasticity. On the other hand, for an auxetic material $2c_2 < c_1 < 3c_2 + c_3$, and thus

$$\sqrt{\frac{1}{2}} < \nu < \sqrt{\frac{3}{4}}.$$

These considerations show that our rotational elasticity model shares many features with well know classical elasticity, however, it also contains many new interesting features.

3.3. Visualising transversal rotational waves assuming planar symmetry.
If we are now separating variables and assume

$$(3.3) \quad \phi_t(x, y, t) = \cos(\omega t)v(x, y)$$

we obtain

$$(3.4) \quad (\Delta + k^2)v = 0, \quad k = \omega \sqrt{\frac{\rho}{c_2 + c_3}}$$

the Helmholtz equation and the dispersion relation for the medium. The Helmholtz equation is a well understood equation within the theory of PDEs. However, we will further simplify our model.

Note that, as expected, the function $\phi \equiv \text{const.}$ solves these equations and hence it is natural to regard this solution as the ground state where all material points of the medium are aligned.

In addition to assuming rotations about the z -axis only and homogeneity along the axis of rotation, we will now also specify a point in the plane orthogonal to the axis of rotation, the $x - y$ plane. This means, we are assuming rotational symmetry in the $x - y$ plane, we write $\phi(x, y, z) = e^{i\omega t}v(r)$ with $r^2 = x^2 + y^2$. The nonlinear problem now reduces to a single ordinary differential equation which is given by

$$(3.5) \quad v''(r) + \frac{1}{r}v'(r) + \frac{\rho\omega^2}{c_2 + c_3}v(r) = 0,$$

and is solved by a linear combination of Bessel functions of first and second kind. We require the solution to regular at $r = 0$, therefore choosing one constant of integration to eliminate the Bessel function of second kind. We are left with the solution

$$(3.6) \quad v(r) = v_0 \mathcal{J}_0 \left(\omega \sqrt{\frac{\rho}{c_2 + c_3}} r \right)$$

where $v_0 = v(0)$ and \mathcal{J}_0 denotes the Bessel function of first kind.

This solution of rotational elasticity can now be visualised in the following manner. Let us attach to every material point in the medium an arrow to indicate its orientation. This is the practical realisation of attaching an orthonormal basis to every point in space. The choice $\phi \equiv 0$ corresponds to the situation where all arrows are parallel to the x -axis, which means that the medium under consideration is assumed to be aligned. We choose $\omega \sqrt{\rho/(c_2 + c_3)} = 1$ and $u_0 = \pi$, where the latter means that we rotate the central material point by a half turn. Thus we investigate the effect of the resulting rotational wave on the rest of the medium. The medium in its ground state $\phi \equiv 0$ is shown in the left panel of Fig. 1 while the distorted medium is shown in the right panel.

4. CONCLUSIONS

A new point of view was introduced to model rotational elasticity in a nonlinear setting using orthogonal matrices as the unknown variables. We showed how this theory can be related to classical elasticity and identified parameter ranges where the rotational medium would correspond to an auxetic material. After identifying

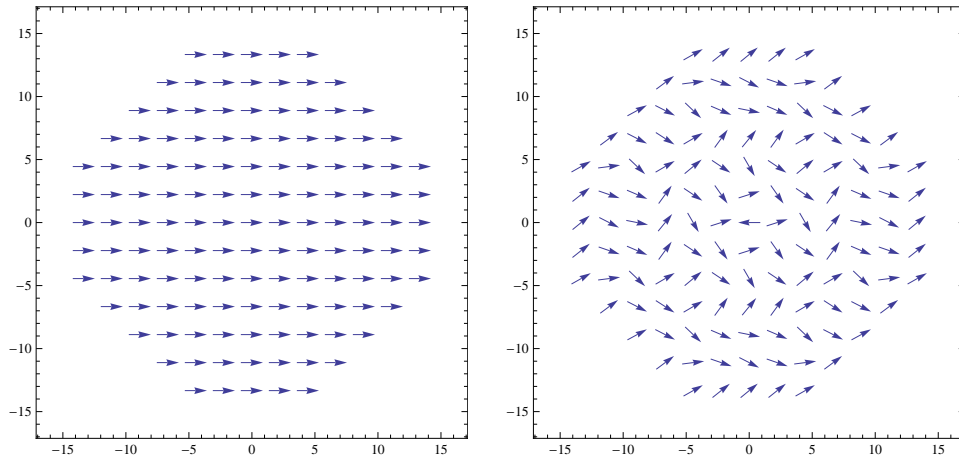


FIGURE 1. Rotational elasticity visualised. The left panel shows the undistorted medium where all material points are aligned parallelly to the x -axis. The right panel depicts the solution of the transversal rotational wave equation when the central material point is rotated by a half turn.

the most general energy functional of this model, two types of plane wave solutions were constructed analytically, they are solutions of the nonlinear Euler-Lagrange equations. These waves correspond to transversal rotational waves and to longitudinal rotational waves. It is interesting that in our rotational setting the transversal wave velocity can be greater than the longitudinal wave velocity. From a PDE point of view this model is very interesting since concepts like coercivity and convexity are difficult to establish due to the variables being matrices. It is therefore natural to consider next the question of existence and uniqueness of the Dirichlet boundary value problem.

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